The renormalization group improved effective potential in massless models

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The effective potential V is considered in massless $\lambda\phi_4^4$ theory. The expansion of V in powers of the coupling λ and of the logarithm of the background field ϕ is reorganized in two ways; first as a series in λ alone, then as a series in $\ln\phi$ alone. By applying the renormalization group (RG) equation to V, these expansions can be summed. Using the condition $V'(v) = 0$ (where v is the vacuum expectation value of ϕ) in conjunction with the expansion of V in powers of $\ln\phi$ fixes V provided $v \neq 0$. In this case, the dependence of V on ϕ drops out and V is not analytic in λ. Massless scalar electrodynamics is considered using the same approach.

I. INTRODUCTION

The effective potential [1, 2, 3, 4] has led to the suggestion that the renormalization induced scale parameter μ might provide a scale parameter for a non vanishing expectation value for a background scalar field ϕ, even if there is no mass parameter in the classical Lagrangian. This was realized in lowest order perturbation theory in the context of massless scalar electrodynamics (MSQED) [1] and further examined in MSQED and the standard model in ref. [5].

The discussion of ref. [5] makes use of the RG equation to sum leading log (LL) contributions to V. The condition $V'(v) = 0$ then serves to relate gauge and scalar self couplings arising in the models considered at a value of $\mu = v$. This was done at one loop order in ref. [1] in MSQED.

In this paper we first consider massless $\lambda\phi^4$. The RG equation when applied to V in this model can be used to determine in closed form LL, NLL (next-to-leading log) etc, contribution to V, much as has been done in refs. [1, 5]. However, it also proves to be convenient to consider an alternative expansion of V in powers of $\ln\phi$. As was shown in the context of the effective action [6], the RG equation serves to fix all contributions to V dependent on $\ln\phi$ in terms of the contribution that is independent of $\ln\phi$. When this is combined with the condition $V'(v) = 0$, this $\ln\phi$-independent piece is fixed and V is determined completely provided $v \neq 0$. This is done either by setting μ equal to v or by treating μ as an independent parameter. Remarkably, the resulting exact expansion for V is not analytic in λ and is independent of ϕ. If one were to avoid these results, it is necessary to have $v = 0$. (We do note though that having V independent of ϕ is consistent with the so called "triviality" [7].) Similar considerations are applied to MSQED. Once more it is found that the RG equation determines all $\ln\phi$ contributions to V in terms of those that are independent of $\ln\phi$. If $V'(v) = 0$ for all μ, then again it follows that V is independent of ϕ provided $v \neq 0$.

II. THE RG EQUATION AND V IN $\lambda\phi^4$

In the massless $\lambda\phi_4^4$ model, one can make the expansion

$$V(\lambda, \phi, \mu) = \sum_{m=0}^{\infty} \sum_{n=0}^{m} T_{m,n} \lambda^{m+1} L^n \phi^4 \tag{1}$$

where $L = \ln(\lambda\phi^2/\mu^2)$. (The dependence of V on $\ln\lambda$ arises when using some variant of the MS renormalization scheme [8]; the counter term approach to renormalization used in [1] absorbs the dependence on $\ln\lambda$ into a finite renormalization.)

We first reorganize the double sum in Eq. (1) so that

$$V(\lambda, \phi, \mu) = \sum_{k=0}^{\infty} S_k(\lambda L) \lambda^{k+1} \phi^4 \tag{2}$$

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where

$$S_k(x) = \sum_{l=0}^{\infty} T_{k+l,l}\, x^l. \tag{3}$$

S_0 is the "LL" sum, S_1 is the "NLL" sum, etc. This sort of reordering of the sum of Eq. (1) has been used by Kastening [9] in conjunction with the effective potential, as well as in a variety of other contexts in refs. [5, 6, 10, 11, 12]. (Other treatments of the effective potential appear in refs. [13, 14, 15, 16].)

Renormalization induces dependence of V on a scale parameter μ that does not occur in the initial Lagrangian. Explicit dependence on μ is compensated for by implicit dependence on μ through the renormalized couplings $\lambda(\mu)$ and the renormalized background field $\phi(\mu)$; we have the RG equation

$$\begin{aligned} &\mu \frac{d}{d\mu} V(\lambda(\mu), \phi(\mu), \mu) = 0 \\ &= \left(\mu \frac{\partial}{\partial \mu} + \beta(\lambda(\mu)) \frac{\partial}{\partial \lambda(\mu)} + \gamma(\lambda(\mu))\, \phi(\mu) \frac{\partial}{\partial \phi(\mu)} \right) V(\lambda(\mu), \phi(\mu), \mu) \end{aligned} \tag{4}$$

where

$$\beta(\lambda(\mu)) = \mu \frac{d\lambda(\mu)}{d\mu} = b_2\, \lambda^2 + b_3 \lambda^3 + \cdots, \tag{5}$$

$$\gamma(\lambda(\mu)) = \frac{\mu}{\phi(\mu)} \frac{d\phi(\mu)}{d\mu} = g_1\, \lambda + g_2 \lambda^2 + \cdots. \tag{6}$$

In ref. [17] it is shown how the coefficients b_i, g_i can be determined by the coefficients $T_{m,n}$ in Eq. (1) by using Eq. (4). Here we wish to consider the inverse problem of how portions of V can be fixed from the RG equation (4) if β and γ are known. One approach would be to use the method of characteristics as in [13, 14, 18, 19]. It is also possible to substitute the expansions of Eqs. (2), (5) and (6) into Eq. (4) and equate coefficients of λ^k to zero. This results in a series of coupled differential equations for $S_k(x)$ with the boundary conditions

$$S_k(0) = T_{k,0}. \tag{7}$$

(This approach has been used in refs. [5, 6, 9, 10, 11, 12].)

The first two of these equations are

$$(-2 + b_2 x) S_0'(x) + (b_2 + 4g_1) S_0(x) = 0 \tag{8}$$

and

$$(-2 + b_2 x) S_1'(x) + (2b_2 + 4g_1) S_1(x) + (b_2 + 2\, g_1 + b_3\, x) S_0' + (b_3 + 4g_2) S_0(x) = 0. \tag{9}$$

Eqs. (8) and (9) have solutions

$$S_0(x) = T_{0,0} \left(1 - \frac{b_2}{2}\, x \right)^{-1-4g_1/b_2} \tag{10}$$

$$S_1(x) = \frac{1}{w^2} \left[T_{1,0} - \left(\frac{b_2}{2} + r \right) \ln(w) - s\,(w-1) \right] \tag{11}$$

(In Eq. (11), we have used the facts that $g_1 = 0$ and $T_{0,0} = 1$ and have set $r = b_3/b_2$, $s = 4g_2/b_2$ and $w = 1 - (b_2/2)x$.) These constitute the LL and NLL contributions respectively to $V(\lambda(\mu), \phi(\mu), \mu)$ provided $x = \lambda(\mu) \ln[\lambda(\mu)\phi^2(\mu)/\mu^2]$. They are analytic in $\lambda(\mu)$ about $\lambda(\mu) = 0$ although there is a "Landau pole" when $w = 0$.

Expanding about $\lambda(\mu) = 0$ yields the one and two loop contributions to V coming from perturbation theory plus all RG accessible portions of higher loop diagrams. Explicit calculation of one-loop diagrams is needed to determine $T_{1,0}$. Subsequent functions $S_n(x)$ which give N^{n}LL contributions to V $(n > 1)$ can be determined from Eq. (4) provided $T_{n,0}$ is known.

There is an alternate to Eq. (2) for reorganizing the double sum in Eq. (1) that has been used in refs. [6, 19]. This is simply to expand in powers of L so that

$$V(\lambda, \phi, \mu) = \sum_{n=0}^{\infty} A_n(\lambda)\, L^n\, \phi^4 \tag{12}$$

where

$$A_n(\lambda) = \sum_{m=n}^{\infty} T_{m,\,n}\, \lambda^{m+1}. \tag{13}$$

Substitution of Eq. (12) into Eq. (4) yields

$$A_{n+1}(\lambda) = \frac{1}{n+1}\left[\frac{1}{2}\,\hat{\beta}(\lambda)\, A_n'(\lambda) + 2\,\hat{\gamma}(\lambda)\, A_n(\lambda)\right] \tag{14}$$

where

$$\hat{\beta} = \frac{\beta}{1-\gamma-\beta/2\lambda}, \quad \hat{\gamma} = \frac{\gamma}{1-\gamma-\beta/2\lambda}. \tag{15}$$

If β and γ are known, Eq. (14) serves to fix A_n $(n > 0)$ in terms of A_0, and hence V itself can be written in terms of A_0. To do this, we follow the approach of refs. [6, 19]. This involves defining

$$B_n(\lambda) = \exp\left(4\int_{\lambda_0}^{\lambda} \frac{\gamma(x)}{\beta(x)}\, dx\right) A_n(\lambda) \tag{16}$$

and setting

$$\eta(\lambda) = \int_{\lambda_0}^{\lambda} \left(\frac{2}{\hat{\beta}(x)}\right) dx \tag{17}$$

so that Eq. (14) is re-expressed as

$$\begin{aligned} B_{n+1}(\lambda(\eta)) &= \frac{1}{n+1}\frac{d}{d\eta} B_n(\lambda(\eta)) \\ &= \frac{1}{(n+1)!}\left(\frac{d}{d\eta}\right)^{n+1} B_0(\lambda(\eta)) \end{aligned} \tag{18}$$

Eq. (12) then becomes, by Eqs. (16) and (18)

$$\begin{aligned} V &= \exp\left(-4\int_{\lambda_0}^{\lambda} \frac{\gamma(x)}{\beta(x)} dx\right) \sum_{n=0}^{\infty} \frac{L^n}{n!}\left(\frac{d}{d\eta}\right)^n B_0(\lambda(\eta))\, \phi^4 \\ &= \exp\left(-4\int_{\lambda_0}^{\lambda} \frac{\gamma(x)}{\beta(x)}\, dx\right) B_0(\lambda(\eta+L))\, \phi^4 \\ &= \exp\left(4\int_{\lambda}^{\lambda(\eta+L)} \frac{\gamma(x)}{\beta(x)}\, dx\right) A_0(\lambda(\eta+L))\, \phi^4. \end{aligned} \tag{19}$$

Now that V has been expressed in terms of A_0, a second condition can be imposed to fix A_0 itself. This condition is

$$\frac{d}{d\phi} V(\lambda, \phi = v, \mu) = 0, \tag{20}$$

where v is an extremum of V. Eqs. (12) and (20) together lead to

$$\sum_{n=0}^{\infty} (4\, A_n(\lambda) + 2\,(n+1)\, A_{n+1}(\lambda)) \left(\ln \frac{\lambda\, v^2}{\mu^2}\right)^n v^3 = 0 \tag{21}$$

The parameter μ^2 is not specified; $\lambda = \lambda(\mu)$ is a function of μ^2 whose evolution is dictated by Eq. (5). In refs. [1, 5] this arbitrariness is exploited in the context of MSQED by setting $\mu^2 = v^2$ in the equation $V'(\phi = v) = 0$ to obtain a relation between the quartic scalar coupling $\lambda(\mu)$ and the gauge coupling $e^2(\mu^2)$ evaluated at this value of μ^2. In ref. [1], V is just the one loop coupling, while in ref. [5] the LL contributions to V are considered. Applying the condition of Eq. (20) when $\mu^2 = v^2$ when V is the one loop does not give a consistent result, as can be seen from the discussions of ref. [1]. However, setting $\mu^2 = \lambda(\mu)\, v^2$ so that all logarithms disappear in Eq. (21) leads to

$$A_1 = -2\, A_0 \tag{22}$$

provided $v \neq 0$ as then only the $n = 0$ term in Eq. (21) survives. As the actual value of $\lambda(\mu^2 = \lambda v^2)$ is an independent quantity, dependent as it is on the constant of integration arising in Eq. (5), Eq. (22) is a functional equation. Together, Eq. (14) with $n = 0$ and Eq.(22) show that

$$A_0'(\lambda) + \left(\frac{4}{\beta(\lambda)} - \frac{2}{\lambda}\right) A_0(\lambda) = 0 \tag{23}$$

whose solution is

$$A_0(\lambda) = K\, \exp\left[-\int_{\lambda_0}^{\lambda} \left(\frac{4}{\beta(x)} - \frac{2}{x}\right) dx\right] \tag{24}$$

where $K = A_0(\lambda_0)$ is a boundary value. If $\beta(\lambda)$ is expanded as in Eq. (5), then Eq. (24) can be rewritten as

$$A_0(\lambda) = K'\, \lambda^{(2+4b_3/b_2^2)}\, \exp\left(\frac{4}{b_2\,\lambda}\right) \exp\left\{-\int_0^{\lambda} \left[\frac{4}{\beta(x)} - 4\left(\frac{1}{b_2\, x^2} - \frac{b_3}{b_2^2 x}\right)\right] dx\right\}. \tag{25}$$

(The divergence appearing in the integral occurring in Eq. (24) when $\lambda_0 \to 0$ has been absorbed into K'.) Together, Eqs. (19) and (24) show that

$$V = K \exp\left(4 \int_{\lambda}^{\lambda(\eta+L)} \frac{\gamma(x)}{\beta(x)}\, dx\right) \exp\left(-\int_{\lambda_0}^{\lambda(\eta+L)} \left(\frac{4}{\beta(x)} - \frac{2}{x}\right) dx\right) \phi^4, \tag{26}$$

which becomes

$$V = A_0(\lambda) \exp\left(\int_{\lambda}^{\lambda(\eta+L)} \left(\frac{4\gamma(x) - 4 + 2\beta(x)/x}{\beta(x)}\right) dx\right) \phi^4. \tag{27}$$

From Eqs. (15), (17) and (27) we obtain

$$\begin{aligned} V &= A_0(\lambda) \exp\left\{-2\left[\eta\left(\lambda(\eta+L)\right) - \eta\right]\right\} \phi^4 \\ &= A_0(\lambda)\, \exp(-2\, L)\, \phi^4. \end{aligned} \tag{28}$$

The definition of L reduces Eq. (28) to

$$V = \frac{A_0(\lambda)\, \mu^4}{\lambda^2}, \tag{29}$$

a result independent of ϕ. It does however satisfy the RG equation (4).

If a counter term renormalization were used to eliminate divergences in the calculation of V, then the logarithm $L = \ln(\lambda\phi^2/\mu^2)$ appearing in Eq. (1) would be replaced by

$$\tilde{L} = \ln\left(\phi^2/\mu^2\right) \tag{30}$$

as occurring in ref. [1]. We would then have

$$V(\lambda, \phi, \mu) = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \tilde{T}_{m,n}\, \lambda^{m+1}\, \tilde{L}^n\, \phi^4 \tag{31}$$

with $T_{m,n}$ and $\tilde{T}_{m,n}$ related by a finite renormalization dependent on $\ln\lambda$. If now

$$\tilde{A}_n(\lambda) = \sum_{m=n}^{\infty} \tilde{T}_{m,n}\,\lambda^{m+1} \tag{32}$$

then Eq. (4) results in

$$\tilde{A}_{n+1}(\lambda) = \frac{1}{n+1}\left[\frac{1}{2}\tilde{\beta}(\lambda)\tilde{A}'_n(\lambda) + 2\tilde{\gamma}(\lambda)\tilde{A}_n(\lambda)\right], \tag{33}$$

an equation similar to Eq. (14), but now we have

$$\tilde{\beta} = \frac{\beta}{1-\gamma}, \quad \tilde{\gamma} = \frac{\gamma}{1-\gamma} \tag{34}$$

in place of $\hat{\beta}$ and $\hat{\gamma}$ of Eq. (15).

Just as Eq. (20) leads to Eq. (22), we find that

$$\tilde{A}_1 = -2\,\tilde{A}_0; \tag{35}$$

together Eqs. (33) and (35) lead to

$$\tilde{A}(\lambda) = \tilde{K}'\,\lambda^{4b_3/b_2^2}\exp\left(\frac{4}{b_2\lambda}\right)\exp\left\{-\int_0^\lambda\left[\frac{4}{\beta(x)} - 4\left(\frac{1}{b_2x^2} - \frac{b_3}{b_2^2x}\right)\right]dx\right\} \tag{36}$$

showing that

$$\tilde{A}_0(\lambda) = \left(\tilde{K}'/K'\right)A_0(\lambda)/\lambda^2 \tag{37}$$

with $A_0(\lambda)$ being given by Eq. (25).

The steps leading to Eq. (19) now result in

$$V = \exp\left(4\int_\lambda^{\lambda(\tilde{\eta}+\tilde{L})}\frac{\gamma(x)}{\beta(x)}dx\right)\tilde{A}_0\left(\lambda(\tilde{\eta}+\tilde{L})\right)\phi^4 \tag{38}$$

where in place of η defined in Eq. (17)

$$\tilde{\eta}(\lambda) = \int_{\lambda_0}^{\lambda}\left(\frac{2}{\tilde{\beta}(x)}\right)dx, \tag{39}$$

so that in analogy to Eq. (28) we arrive at

$$V = \tilde{A}_0\exp\left(-2\tilde{L}\right)\phi^4 = \tilde{A}_0(\lambda)\,\mu^4. \tag{40}$$

once again, V is independent of ϕ. From Eq. (37) we see that the dependency of V on λ in Eqs. (29) and (40) is identical.

It is worth noting at this point that the renormalized coupling defined in ref. [1] to be $V''''(\phi=\mu)$ is zero if V is independent of ϕ, as is the case with V given by Eq. (29).

The only way to circumvent this conclusion that V reduces to a non-analytic function of λ which is independent of ϕ is to take the solution to Eq. (21) to be

$$v = 0 \tag{41}$$

rather than have Eq. (22) satisfied. This implies that the symmetry $\phi \leftrightarrow -\phi$ is unbroken by radiative corrections and that ϕ remains massless. The functions $A_n(\lambda)$ $(n>0)$ are still determined in terms of $A_0(\lambda)$ by Eq. (14), but $A_0(\lambda)$ is itself not fixed as Eq. (22) cannot be invoked.

If Eq. (21) were to hold for all μ, then it is satisfied order-by-order in $\ln(\lambda v^2/\mu^2)$ provided

$$A_{n+1}(\lambda) = -\frac{2}{n+1}A_n(\lambda). \tag{42}$$

Eq. (42) reduces to Eq. (22) when $n = 0$; however now we find that Eqs. (14) and (42) give rise to an equation for all A_n, not just A_0. Eq. (24) generalizes now to

$$A_n(\lambda) = K_n \exp\left[-\int_{\lambda_0}^{\lambda}\left(\frac{4}{\beta(x)} - \frac{2}{x}\right) dx\right] \tag{43}$$

with $K_{n+1} = -2K_n/(n+1)$ in order for Eq. (42) be satisfied. Furthermore, from Eq. (42) by itself,

$$A_n(\lambda) = \frac{(-2)^n}{n!} A_0(\lambda) \tag{44}$$

so that

$$V = \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} A_0(\lambda)\left(\ln\frac{\lambda\phi^2}{\mu^2}\right)^n \phi^4 \tag{45}$$

which reproduces the result of Eq. (29).

The approach of this section can be easily adapted to the effective potential in a massless $\lambda\phi_3^6$ model. In analogy with Eq. (29), we find that V is independent of the background field and non-analytic in the coupling λ at $\lambda = 0$.

III. THE RG EQUATION AND V IN MSQED

The LL approximation to V in MSQED has been considered in ref. [5]. Here we consider the consequence of making an expansion of V in MSQED that is similar to Eq. (12),

$$V(\lambda, e^2, \phi, \mu) = \sum_{n=0}^{\infty} A_n(\lambda, e^2)\, L^n\, \phi^4 \tag{46}$$

where again $L = \ln(\lambda\,\phi^2/\mu^2)$. The RG equation now becomes

$$\left(\mu\frac{\partial}{\partial\mu} + \beta_\lambda(\lambda, e^2)\frac{\partial}{\partial\lambda} + \beta_{e^2}(\lambda, e^2)\frac{\partial}{\partial e^2} + \gamma(\lambda, e^2)\phi\frac{\partial}{\partial\phi}\right) V = 0. \tag{47}$$

Just as Eq. (14) was derived, we find now from Eq. (47) that

$$A_{n+1} = \frac{1}{n+1}\left(\frac{1}{2}\hat{\beta}_\lambda\frac{\partial}{\partial\lambda} + \frac{1}{2}\hat{\beta}_{e^2}\frac{\partial}{\partial e^2} + 2\,\hat{\gamma}\right) A_n \tag{48}$$

where

$$\hat{\beta}_\lambda = \frac{\beta_\lambda}{1-\gamma-\beta_\lambda/2\lambda},\quad \hat{\beta}_{e^2} = \frac{\beta_{e^2}}{1-\gamma-\beta_\lambda/2\lambda},\quad \hat{\gamma} = \frac{\gamma}{1-\gamma-\beta_\lambda/2\lambda}. \tag{49}$$

From the analogue of Eq. (20)

$$\frac{d}{d\phi} V(\lambda, e^2, \phi = v, \mu) = 0, \tag{50}$$

we find that, much like Eq. (21),

$$\sum_{n=0}^{\infty}(4\,A_n(\lambda, e^2) + 2\,(n+1)\,A_{n+1}(\lambda, e^2))\left(\ln\frac{\lambda\, v^2}{\mu^2}\right)^n v^3 = 0. \tag{51}$$

If $v \neq 0$ and Eq. (51) holds order-by-order in $\ln(\lambda v^2/\mu^2)$, then much like Eq. (42) we have

$$A_{n+1}(\lambda, e^2) = -\frac{2}{n+1} A_n(\lambda, e^2) \tag{52}$$

so that

$$A_{n+1}(\lambda, e^2) = \frac{(-2)^{n+1}}{(n+1)!}\, A_0(\lambda, e^2). \tag{53}$$

Together, Eqs (46) and (53) show that

$$V(\lambda, e^2, \phi, \mu) = \frac{A_0(\lambda, e^2)\, \mu^4}{\lambda^2}; \tag{54}$$

Eq. (54) is the generalization of eq. (29) to MSQED. Again, dependence on ϕ has disappeared and the dependence on λ is not analytic at $\lambda = 0$. This situation is avoided only if $v = 0$ is taken to be the solution to Eq. (51), in which case there is not spontaneous symmetry breakdown. We note that even with the flat effective potential of Eq. (54), the model is not trivial and it is possible to have $v \neq 0$ in which case the gauge field develops a mass.

In refs. [1, 5] Eq. (50) is only applied when $\mu^2 = \lambda(\mu)v^2$. At one loop (cf. ref. [1]) or LL (cf. ref [5]) level, this results in a relationship between λ and e^2 at this mass scale. More generally, from Eq. (50) we find that when $\mu^2 = \lambda(\mu)v^2$

$$A_1(\lambda, e^2) = -2A_0(\lambda, e^2) \tag{55}$$

the $n = 0$ limit of Eq. (52), again provided $v \neq 0$. The numerical values of λ and e^2 can be independently varied as they are contingent upon the boundary values for the functions $\lambda(\mu)$, $e^2(\mu)$ and hence Eq. (55) is a functional relation for all λ, e^2.

We will now show that Eqs. (48) and (55) are sufficient to establish Eq. (54), much as Eqs. (14) and (22) are shown to lead to Eq. (29). Our method reduces to that outlined in Eqs. (12) to (29) above in the limit $e^2 = 0$. It involves adapting the "method of characteristics" (refs. [13, 14, 18, 19]) to the case in hand.

We begin by noting that from Eq. (48) with $n = 0$ and Eq. (55) that $A_0(\lambda, e^2)$ must satisfy

$$\left[\frac{1}{2}\,\hat{\beta}_\lambda \frac{\partial}{\partial\lambda} + \frac{1}{2}\,\hat{\beta}_{e^2} \frac{\partial}{\partial e^2} + 2\,(1+\hat{\gamma})\right] A_0(\lambda, e^2) = 0. \tag{56}$$

Characteristic functions $\bar{\lambda}(t)$ and $\bar{e}^2(t)$ are now defined by

$$\frac{d\bar{\lambda}(t)}{dt} = \frac{1}{2}\hat{\beta}_\lambda(\bar{\lambda}(t), \bar{e}^2(t)), \quad \bar{\lambda}(0) = \lambda \tag{57}$$

$$\frac{d\bar{e}^2(t)}{dt} = \frac{1}{2}\hat{\beta}_{e^2}(\bar{e}^2(t), \bar{e}^2(t)), \quad \bar{e}^2(0) = e^2 \tag{58}$$

We set

$$a_n(\bar{\lambda}(t), \bar{e}^2(t), t) = \exp\left\{2\int_0^t \hat{\gamma}(\bar{\lambda}(\tau), \bar{e}^2(\tau))d\tau\right\} A_n(\bar{\lambda}(t), \bar{e}^2(t)) \tag{59}$$

so that

$$\begin{aligned}\frac{d}{dt}a_n(\bar{\lambda}(t), \bar{e}^2(t), t) \;&=\; \exp\left\{2\int_0^t \hat{\gamma}(\bar{\lambda}(\tau), \bar{e}^2(\tau))d\tau\right\} \\ &\left[\frac{1}{2}\,\hat{\beta}_\lambda(\bar{\lambda}, \bar{e}^2)\frac{\partial}{\partial\bar{\lambda}} + \frac{1}{2}\,\hat{\beta}_{e^2}(\bar{\lambda}, \bar{e}^2)\frac{\partial}{\partial\bar{e}^2} + 2\hat{\gamma}(\bar{\lambda}, \bar{e}^2)\right] A_n(\bar{\lambda}, \bar{e}^2).\end{aligned} \tag{60}$$

From Eqs. (48), (59) and (60), we find that

$$\frac{d}{dt}a_n(\bar{\lambda}(t), \bar{e}^2(t), t) = (n+1)a_{n+1}(\bar{\lambda}(t), \bar{e}^2(t), t). \tag{61}$$

Furthermore, we define

$$V(t) = \sum_{n=0}^{\infty} a_n(\bar{\lambda}(t), \bar{e}^2(t), t)\bar{L}^n \phi^4 \tag{62}$$

where

$$\frac{d\phi(t)}{dt} = \frac{1}{2}\hat{\gamma}(\bar{\lambda}(t), \bar{e}^2(t))\bar{\phi}(t), \quad \bar{\phi}(0) = \phi \tag{63}$$

and

$$\frac{d\bar{\mu}(t)}{dt} = \frac{1}{2}\frac{\bar{\mu}(t)}{1 - \gamma(\bar{\lambda}(t), \bar{e}^2(t)) - \beta_\lambda(\bar{\lambda}(t), \bar{e}^2(t))/2\bar{\lambda}(t)}, \quad \bar{\mu}(0) = \mu \tag{64}$$

with

$$\bar{L} = \ln\left(\frac{\bar{\lambda}(t)\bar{\phi}^2(t)}{\bar{\mu}^2(t)}\right). \tag{65}$$

By Eqs. (59) and (60), it is apparent that

$$\frac{dV(t)}{dt} = 0 \tag{66}$$

and

$$V(0) = V(\lambda, e^2, \phi, \mu^2). \tag{67}$$

By Eq. (61) we see that

$$a_n(\bar{\lambda}(t), \bar{e}^2(t), t) = \frac{1}{n!}\left(\frac{d}{dt}\right)^n a_0(\bar{\lambda}(t), \bar{e}^2(t), t) \tag{68}$$

so that Eq. (62) becomes

$$V(t) = \sum_0^\infty \frac{\bar{L}^n}{n!}\left(\frac{d}{dt}\right)^n a_0(\bar{\lambda}(t), \bar{e}^2(t), t)\phi^4 \tag{69}$$

or

$$V(t) = a_0(\bar{\lambda}(t + \bar{L}, \bar{e}^2(t + \bar{L}), t + \bar{L})\phi^4 \tag{70}$$

If we set

$$\tilde{A}_0(\bar{\lambda}(t), \bar{e}^2(t), t) = \exp\left\{2\int_0^t \left[1 + \hat{\gamma}(\bar{\lambda}(\tau), \bar{e}^2(\tau), \tau)\right] d\tau\right\} A_0(\bar{\lambda}(t), \bar{e}^2(t)) \tag{71}$$

then by Eq. (56)

$$\frac{d}{dt}\tilde{A}_0(\bar{\lambda}(t), \bar{e}^2(t), t) = 0 \tag{72}$$

with

$$\tilde{A}_0(\bar{\lambda}(0), \bar{e}^2(0), 0) = A_0(\lambda, e^2). \tag{73}$$

Together, Eqs. (59) and (71) imply that

$$a_0(\bar{\lambda}(t), \bar{e}^2(t), t) = \exp(-2t)\tilde{A}_0(\bar{\lambda}(t), \bar{e}^2(t), t) \tag{74}$$

so that Eq. (70) becomes

$$\begin{aligned} V(t) &= \exp\left(-2(t + \bar{L})\right)\tilde{A}_0(\bar{\lambda}(t + \bar{L}), \bar{e}^2(t + \bar{L}), t + \bar{L})\phi^4 \\ &= \exp(-2t)\frac{\tilde{A}_0(\bar{\lambda}(t + \bar{L}), \bar{e}^2(t + \bar{L}), t + \bar{L})\bar{\mu}^4(t)\phi^4}{\bar{\lambda}^2(t)\bar{\phi}^4(t)}. \end{aligned} \tag{75}$$

From Eqs. (66), (67) and (75) we obtain

$$V(\lambda, e^2, \phi, \mu) = \frac{\tilde{A}_0(\bar{\lambda}(L), \bar{e}^2(L), L)\mu^4}{\lambda^2}, \tag{76}$$

which when combined with Eqs. (72) and (73) gives

$$V(\lambda, e^2, \phi, \mu) = \frac{A_0(\lambda, e^2)\mu^4}{\lambda^2}. \tag{77}$$

Eq. (54) is consequently recovered using Eqs. (48) and (55) alone.

IV. DISCUSSION

We have considered the effective potential V, first in the massless $\lambda\phi^4$ model. The RG equation has been used to generate a closed form expression for all LL contributions to V from the one loop result; in general all N^{n}LL contributions can be found provided the n-loop contribution to V has been computed.

In addition, it has been possible to express all log dependent contributions to V in terms of the log independent piece. When the RG equation is supplemented by $V'(\phi = v) = 0$, then if $v \neq 0$, this log independent piece of V has been calculated, showing V to be independent of ϕ and non-analytic in λ at $\lambda = 0$.

These results are then shown to also hold in the MSQED model. It is likely that the arguments apply in arbitrary massless models in which scalars couple to gauge Bosons and Fermions.

By having included all log dependent contributions to V in these two models and summing them through application of the RG equation and the condition $V'(\phi = v) = 0$, we have come to a result at odds with the scenario envisioned in ref. [1] for massless models involving scalars; $V(\phi)$ is now flat, having neither a local minimum nor a singularity associated with a "Landau pole". The discrepancy arises as a result of the way the condition of Eq. (20) has been used. For us it implies a relationship between A_1 and A_0 (given by Eqs. (22) and (55)), where A_0 and A_1 are respectively the contributions to V that are log independent and leading log to all orders. In ref. [1], these conditions are applied to the lowest order perturbative contribution to V. This implies in the massless ϕ_4^4 model that the minimum v lies outside the perturbative range in the expansion for V as $\lambda \ln(v^2/\mu^2)$ is exceedingly large. For MSQED it is used to imply a relationship between the quartic scalar coupling and the electric charge, evaluated at a mass scale equal to the expectation value of v. This is a one loop limit of Eq. (55). By using the all orders (rather than the one loop order) contributions $A_0(\lambda, e^2)$ and $A_1(\lambda, e^2)$ we are able to relate these two functions at this mass scale; rather than arguing that this fixes λ in terms of e^2, we take these couplings to have independent values and the functions A_0 and A_1 to be dependent on each other according to Eq. (55), allowing us to eventually demonstrate that V is independent of ϕ if $v \neq 0$.

The effect on this analysis of including mass parameters into the initial classical Lagrangian is clearly an outstanding problem that needs to be addressed.

Acknowledgments

We would like to thank A. Buchel, V. Elias, E. V. Gorbar, R. B. Mann, V. A. Miransky, A. Rebhan, T. N. Sherry and T. G. Steele for helpful discussion. NSERC and Fapesp provided financial support. DGCM would like to thank Perimeter Institute for Theoretical Physics for its hospitality where part of this work was done and FTB would like to thank the Department of Applied Mathematics of The University of Western Ontario for the its hospitality. R. and D. MacKenzie had a useful suggestion.

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